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THE P-VERSION OF THE FINITE ELEMENT METHOD FOR  
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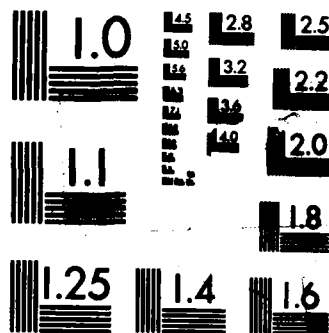
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The p-version of the finite element method  
for constraint boundary conditions

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The p-version of the finite element method  
for constraint boundary conditions

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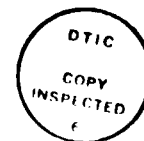
The p-version of the finite element method  
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→ This

**Abstract.** The paper addresses the implementation of general constraint boundary conditions for a system of equations by the p-version of the finite element method. By constraint boundary conditions <sup>the authors</sup> we mean conditions where some relation between the components is prescribed at the boundary. Optimal error bounds are proven.

*Keywords: convergence; Sobolev spaces.* ←

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## 1. Introduction.

There is a large variety of boundary conditions for systems of differential equations of elliptic type. Some physically natural conditions may be formulated by a variational approach through constraint conditions. For example, the two dimensional elasticity problem can be formulated as the minimization of a quadratic functional  $F(u)$ ,  $u = (u_1, u_2)$  over a set  $H$  satisfying

$$(H_0^1(\Omega))^2 \subset H \subset (H^1(\Omega))^2.$$

Selections of  $H$  then characterize the boundary conditions.

Obviously the choice  $H = (H_0^1(\Omega))^2$  induces the (essential) Dirichlet conditions, i.e., the displacement is given on  $\partial\Omega$ , while  $H = (H^1(\Omega))^2$  induces the (natural) Neumann conditions, i.e., the tractions are prescribed on  $\partial\Omega$ . In addition to these classical conditions other types are important in applications. One of these conditions is characterized by

$$(1.1) \quad H = \{(u_1, u_2) \in (H^1(\Omega))^2 \mid u_1 \varphi_1(s) + u_2 \varphi_2(s) = 0 \text{ on } \partial\Omega\}$$

where  $\varphi_1$  and  $\varphi_2$  are given functions defined on  $\partial\Omega$ . These conditions are in the most simple case the symmetry conditions and in general traction free constraints at the boundary.

So far we have only mentioned homogeneous boundary conditions. Nonhomogeneous conditions are defined in the usual way, when the minimization of  $F$  is over a hyperplane  $H_v = \{u + v \mid u \in H, v \in (H^1(\Omega))^2\}$ .

The constraint boundary condition we mentioned above is a type of essential condition. Hence when solving such problems by the finite element method in general and by the  $p$  or  $h$ - $p$  versions

in particular, we face the problem of implementing the nonhomogeneous boundary conditions (which are outside the finite element space).

The  $p$  and  $h$ - $p$  versions are recent developments, where  $p$ , the degree of the elements used is not fixed but is increasing. This is in contrast to the classical  $h$ -version, where the degree  $p$  is kept fixed. The first commercial programs available are PROBE (Noetic Tech., St. Louis) and FIESTA (ISMES, Bergamo, Italy).

The implementation of Dirichlet boundary conditions for the  $p$ -version of the finite element method has been addressed by us in [2] and [4]. A general survey on the state of the art of the  $p$  and  $h$ - $p$  versions may be found in [1].

In this paper we will address the implementation of the constraint conditions (1.1) in a simplified setting (to avoid notational difficulties). Section 2 deals with preliminaries and notation. In Section 3 we formulate an abstract approach and based on it prove that the suggested finite element formulation of the constraint boundary condition leads to the optimal rate of convergence of the  $p$ -version. Section 4 addresses some implementational aspects.



## 2. The basic notation and preliminaries.

### 2.1. The Sobolev spaces

Let  $\mathbb{R}^2$  be the two dimensional Euclidean space,  $x = (x_1, x_2) \in \mathbb{R}^2$ . Let  $\Omega \subset \mathbb{R}^2$  be a bounded Lipschitzian domain with the boundary  $\Gamma = \partial\Omega$ . We will assume that  $\Gamma$  is a Jordan curve,  $\Gamma = \bigcup_{i=1}^m \bar{\Gamma}_i$  where  $\Gamma_i$  are smooth open arcs with parametric description

$$\Gamma_i = \{(x_1, x_2) | x_1 = x_{1,1}(\xi), x_2 = x_{1,2}(\xi), |\xi| < 1\},$$

$$i = 1, \dots, m.$$

Denoting  $I = (-1, 1)$ ,  $\Gamma_i$  is obviously the image of  $I$  by the mapping  $F_i = \{x_{1,1}, x_{1,2}\}$ , i.e.,  $\Gamma_i = F_i(I)$ . If  $u(s)$  is defined on  $\Gamma_i$  then by  $U(\xi) = u(F_i(\xi))$  we denote its transform on  $I$ . The ends of  $\Gamma_i$  will be called vertices and denoted by  $A_i = (x_{1,1}(-1), x_{1,2}(-1))$ ,  $B_i = (x_{1,1}(1), x_{1,2}(1))$ . We will further assume that  $B_i = A_{i+1}$ ,  $B_m = A_1$ ,  $i = 1, \dots, m$ . By this, the orientation of  $\Gamma_i$  is established. In general we will denote the vertices by  $A_i (= B_{i-1})$ ,  $i = 1, \dots, m$ . The scheme of the domain and the pertinent notation is shown in Figure 2.1.

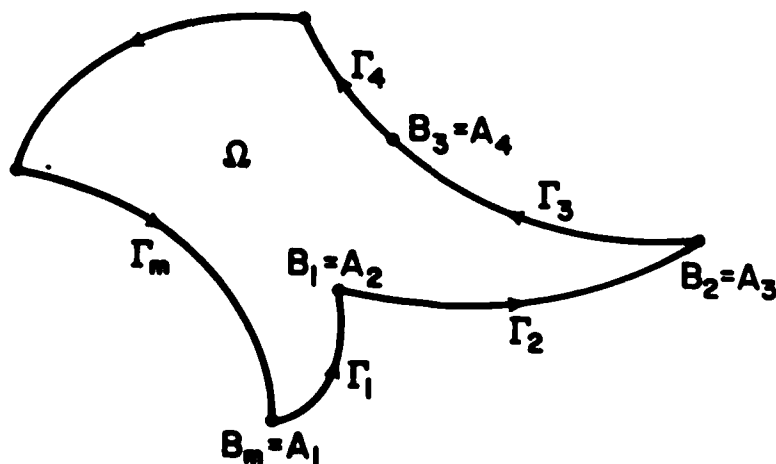


Figure 2.1. Scheme of the domain and notation.

Remark 2.1. We assumed that the domain  $\Omega$  is simply connected. This assumption has been made only for notational simplicity.

Remark 2.2. We assumed that the domain is Lipschitzian. Once more, our results are valid (with proper modification) in the case when, for example, some arcs coincide (as in the case of the slit domain).

Remark 2.3. We have assumed that the arcs  $\Gamma_i$  are sufficiently smooth. For the sake of simplicity we assume that they are  $C^\infty$  arcs (i.e., the functions  $x_{i,j}$ ,  $i = 1, \dots, m$ ,  $j = 1, 2$  are  $C^\infty$  functions).

By  $H^k(\Omega)$ ,  $k \geq 0$  integer we denote the usual Sobolev space of functions with square integrable derivatives on  $\Omega$ . The norm will be denoted by  $\|\cdot\|_{H^k(\Omega)}$ . If  $\ell < q < \ell+1$ ,  $\ell \geq 0$  integer, then we define  $H^q(\Omega) = (H^\ell(\Omega), H^{\ell+1}(\Omega))_\theta$ ,  $\theta = q - \ell$  where by  $(\cdot, \cdot)_\theta$  we denote the usual interpolated space using the K-method (see [5]). The scalar product  $(\cdot, \cdot)_{H^q}$  and the norm  $\|\cdot\|_{H^q(\Omega)}$  are defined accordingly.

By  $C^k(\bar{\Omega})$ ,  $k \geq 0$  integer, we denote the space of all functions with  $k$  continuous derivatives on  $\bar{\Omega}$ . It is possible to show that  $H^k(\Omega) \hookrightarrow C^0(\bar{\Omega})$  for  $k > 1$ , where by  $\hookrightarrow$  we denote continuous imbedding. On the other hand,  $H^1(\Omega) \not\subset C^0(\bar{\Omega})$ .

For  $I = (-1, 1)$ ,  $H^k(I)$ ,  $k \geq 0$  is defined analogously as before. If  $k > 1/2$  then  $H^k(I) \hookrightarrow C^0(\bar{I})$  but  $H^k(I) \not\subset C^0(\bar{I})$  for  $k \leq 1/2$ .

So far we have defined  $H^k(I)$ ,  $k \geq 0$ . We will also be interested in  $H^k(I)$ ,  $k < 0$ . We define for  $k \geq 0$

$$\|u\|_{H^{-k}(I)} = \sup_{\substack{v \neq 0 \\ v \in H^k(I)}} \frac{\int_{-1}^1 u v dx}{\|v\|_{H^k(I)}}.$$

(Let us remark that sometimes (see e.g. [5]) our space  $H^{-k}(I)$  is denoted by  $(H^k(I))'$  whereas  $H^{-k}(I)$  is used to denote the dual space of  $H_0^k(I)$ ).

If  $u$  is defined on  $\Gamma_1$  then we define

$$H^k(\Gamma_1) = \{u | u(F_1(\xi)) = U(\xi) \in H^k(I)\}$$

$$\|u\|_{H^k(\Gamma_1)} = \|U\|_{H^k(I)}$$

So far we have considered only scalar functions on  $\Omega$  and  $I$ .

The spaces of vector functions are defined by Cartesian products,

$${}^2H^k(\Omega) = (H^k(\Omega))^2.$$

Let now

$$Q = \{(x_1, x_2) | |x_1| < 1, |x_2| < 1\}$$

$$\gamma_1^Q = \{(x_1, x_2) | |x_1| < 1, x_2 = -1\}$$

$Q$  will be called the standard square and  $\gamma_1^Q$ ,  $i = 1, 2, 3, 4$  its sides ( $\gamma_i^Q$ ,  $i = 2, 3, 4$  are defined analogously to  $\gamma_1^Q$  in an obvious way). Let

$$T = \{(x_1, x_2) | |x_1| < 1, 0 < x_2 < (1+x_1)\sqrt{3} \text{ for } x_1 < 0,$$

$$0 < x_2 < (1-x_1)\sqrt{3} \text{ for } x_1 > 0\}$$

$$\gamma_1^T = \{(x_1, x_2) | |x_1| < 1, x_2 = 0\}.$$

$T$  will be called the standard triangle and  $\gamma_i^T$ ,  $i = 1, 2, 3$  its sides.

Let us remark that the sides of  $T$  and  $Q$  are each of length 2. Later we will often not distinguish between  $\gamma_1$  and  $I$ .

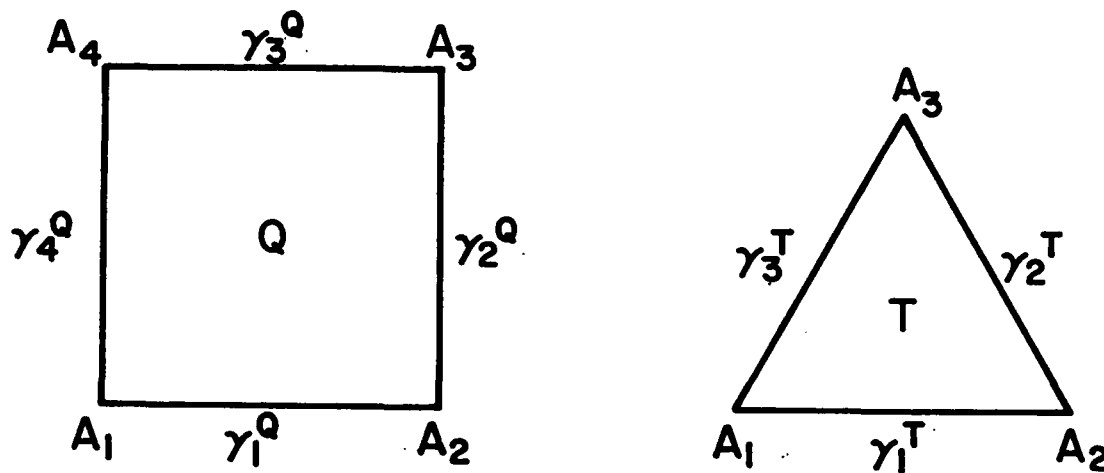


Figure 2.2. The scheme of the standard square and standard triangle.

We now define

$$\mathcal{P}_p^2(Q) = \{u \mid u \text{ is a polynomial of degree } \leq p \text{ in each variable } x_1 \text{ and } x_2 \text{ over } Q\}.$$

$$\mathcal{P}_p^1(T) = \{u \mid u \text{ is a polynomial of (total) degree } \leq p \text{ on } T\}.$$

$$\mathcal{P}_p(I) = \{u \mid u \text{ is a polynomial of degree } \leq p \text{ on } I\}.$$

We have then

**Lemma 2.1.** Let  $v \in H_0^1(\gamma_1^Q) \cap \mathcal{P}_p(\gamma_1^Q)$  (respectively  $v \in H_0^1(\gamma_1^T) \cap \mathcal{P}_p(\gamma_1^T)$ ) such that

$$\|v\|_{H^t(\gamma_1^Q)} \leq p^{-(1-t)} A, \quad t = 0, 1$$

respectively

$$\|v\|_{H^t(\gamma_1^T)} \leq p^{-(1-t)} A, \quad t = 0, 1.$$

Then there exists  $u \in \mathcal{P}_p^2(Q)$  (respectively  $\mathcal{P}_p^1(T)$ ) such that  $u|_{\gamma_1^Q} = v$  (respectively  $u|_{\gamma_1^T} = v$ ),  $u|_{\partial Q - \gamma_1^Q} = 0$  (respectively  $u|_{\partial T - \gamma_1^T} = 0$ ) and

$$\|u\|_{H^1(Q)} \leq C p^{-1/2} A$$

respectively

$$\|u\|_{H^1(T)} \leq C p^{-1/2} A.$$

For the proof see [2] or [3].

## 2.2. The model problem

Let

$${}^2H_0^1(Q) \subset \mathcal{X}(Q) \subset {}^2H^1(Q)$$

where  $\mathcal{X}(Q)$  is closed in  ${}^2H^1(Q)$ .  $\mathcal{X}(Q)$  will be called the constraint space. Assume that there is given a continuous bilinear form  $B(u, v)$  on  ${}^2H^1(Q) \times {}^2H^1(Q)$ ,  $u = (u_1, u_2)$ ,  $v = (v_1, v_2)$  such that

$$(2.1) \quad B(u, u) \geq \gamma \|u\|_{{}^2H^1(Q)}^2, \quad \gamma > 0 \quad \text{for any } u \in {}^2H^1(Q).$$

Then obviously for any  $G_1 \in (\mathcal{X}(Q))'$ , there is a unique  $u_0 \in \mathcal{X}(Q)$  such that

$$B(u_0, v) = G_1(v)$$

holds for any  $v \in \mathcal{X}(Q)$ . We also have

$$\|u_0\|_{{}^2H^1(Q)} \leq C \|G_1\|_{({}^2H^1(Q))'}$$

Denote  $\mathcal{K}_\rho(\Omega) = \{u \in {}^2H^1(\Omega), u - \rho \in \mathcal{K}(\Omega)\}$ .  $\mathcal{K}_\rho(\Omega)$  will be called the  $\rho$ -hyperplane. Then our model problem is given by: Find  $u_0 \in \mathcal{K}_\rho(\Omega)$  such that

$$(2.2) \quad B(u_0, v) = G_1(v), \quad \forall v \in \mathcal{K}(\Omega).$$

We have then

$$(2.3) \quad \|u_0\|_{{}^2H^1(\Omega)} \leq C[\|\rho\|_{{}^2H^1(\Omega)} + \|G_1\|_{{}^2H^1(\Omega)}].$$

If  $\rho = 0$  then we will speak about a homogeneous constraint problem while for  $\rho \neq 0$  we will speak about a nonhomogeneous constraint problem. We call these constraint problems because  $\mathcal{K}(\Omega) \neq {}^2H^1(\Omega)$ .

There are many constraint problems in applications. We will consider the one when

$$\mathcal{K}(\Omega) = \{(u_1, u_2) \in {}^2H^1(\Omega) \mid \sum_{\ell=1}^2 \alpha_{k,\ell}^{(j)} u_\ell|_{\Gamma_j} = 0, k = 1, 2, j = 1, \dots, m\}$$

where  $\alpha^j = \{\alpha_{k,\ell}^{(j)}\}$  are matrices of smooth functions on  $\bar{\Gamma}_j$  (say  $C^\infty(\bar{\Gamma}_j)$ ). Additional assumptions on  $\{\alpha_{k,\ell}^{(j)}\}$  will be imposed later.

Obviously when  $\alpha_{k,k} = 1, \alpha_{k,\ell} = 0$  for  $k \neq \ell$  we get Dirichlet boundary conditions (in general we get Dirichlet conditions when  $\alpha^j$  have rank 2 for all  $x \in \Gamma_j$ ). If  $\{\alpha_{k,\ell}^{(j)}\} = 0$  there is no constraint and we have the Neumann problem.

If  $\alpha^j$  has rank 1 then we can write the constraint on  $\Gamma_j$  as

$$\alpha_{1,1}^{(j)} u_1 + \alpha_{1,2}^{(j)} u_2 = 0$$

which will be written in the form

$$(2.4a) \quad \alpha^{(j)} u_1 + \beta^{(j)} u_2 = 0.$$

Obviously if  $\rho = (\rho_1, \rho_2)$  then the nonhomogeneous constraint problem is characterized on  $\Gamma_j$  by

$$(2.4b) \quad \alpha^{(j)} u_1 + \beta^{(j)} u_2 = \alpha^{(j)} \rho_1 + \beta^{(j)} \rho_2.$$

Problems of this type are common, for example, in the theory of elasticity. For simplicity of the exposition and notation we will restrict ourselves to the model problem where

$$(2.5) \quad B(u, v) = \int_{\Omega} \left( \frac{\partial u_1}{\partial x_1} \frac{\partial v_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \frac{\partial v_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \frac{\partial v_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial v_2}{\partial x_2} + u_1 v_1 + u_2 v_2 \right) dx_1 dx_2$$

and

$$G_1(v) = \int_{\Omega} (f_1 v_1 + f_2 v_2) dx_1 dx_2, \quad f = (f_1, f_2) \in {}^2H^0(\Omega).$$

Although we restrict ourselves to this special case, our results hold in general, e.g., for elasticity problems, etc.

We will assume that  $a_{k,\ell}^{(j)} \in C^x(\bar{\Gamma}_j)$ . In practice we have the nonhomogeneous constraint problem defined so that

1) If  $\{a_{k,\ell}^{(j)}\}$  has rank 2 on  $\bar{\Gamma}_j$  then the constraint is

$$\sum_{\ell=1}^2 a_{k,\ell}^{(j)} u_{\ell} = g_k^{(j)}, \quad k = 1, 2$$

where  $(g_1^{(j)}, g_2^{(j)})$  are defined on  $\bar{\Gamma}_j$ . Hence obviously we can transform the above constraint equation to

$$(2.6) \quad \begin{aligned} u_1 &= \tilde{g}_1^{(j)} \\ u_2 &= \tilde{g}_2^{(j)}. \end{aligned}$$

Because  $\alpha_{k,\ell}^{(j)}$  are assumed to be smooth, we see that  $\{\bar{g}_i^{(j)}\}$ ,  $i = 1, 2$  have the same smoothness as  $\{g_i^{(j)}\}$ .

ii) If  $\{\alpha_{k,\ell}^{(j)}\}$  has rank 1 then the constraint equation is

$$\alpha^{(j)} u_1 + \beta^{(j)} u_2 = g^{(j)}.$$

We add the condition

$$\alpha^{(j)} + \beta^{(j)} = 0 \text{ on } \bar{\Gamma}_j.$$

This enables us to transform the constraint equation to

$$(2.7) \quad \alpha^{(j)} u_1 + \beta^{(j)} u_2 = \tilde{g}^{(j)} \text{ with } \alpha^{(j)} + \beta^{(j)} = 1.$$

As formulated above,  $\tilde{g}^{(j)}$  are defined separately on each  $\bar{\Gamma}_j$ . We will assume that  $\tilde{g}^{(j)}$  satisfy consistency conditions, namely that there exists  $\rho = (\rho_1, \rho_2) \in {}^2H^1(\Omega)$  such that

$$(2.8a) \quad \tilde{g}_1^{(j)} = \rho_1|_{\bar{\Gamma}_j}, \quad \tilde{g}_2^{(j)} = \rho_2|_{\bar{\Gamma}_j},$$

respectively

$$(2.8b) \quad \alpha^{(j)} \rho_1|_{\bar{\Gamma}_j} + \beta^{(j)} \rho_2|_{\bar{\Gamma}_j} = \tilde{g}^{(j)}.$$

These conditions have to be imposed especially at the vertices of  $\Omega$ .

The sides  $\Gamma_j$  where the constraint (2.6) is imposed will be called total constraint sides, while  $\bar{\Gamma}_j$  where the constraint (2.7) is imposed will be called partial constraint sides. We will enumerate the total constraint sides as  $\Gamma_{i_j}$ ,  $j = 1, \dots, m_1$  and the partial constraint sides as  $\bar{\Gamma}_{i_j}$ ,  $j = m_1 + 1, \dots, m$ .

### 2.3. The p-version of the finite element method

Assume that the domain  $\Omega$  has been partitioned into a finite



number of subdomains  $\Omega_i$ , i.e.,  $\bar{\Omega} = \bigcup_{i=1}^n \bar{\Omega}_i$ . We shall assume that  $\Omega_i$  is the curvilinear quadrilateral

$$\Omega_i = \tau_i(Q)$$

or curvilinear triangle

$$\Omega_i = \tau_i(T)$$

where  $Q$  and  $T$  are the standard square and triangle, respectively. The domains  $\Omega_i$  will be called elements. We will assume that  $\tau_i^{-1}$  is a smooth one to one mapping of  $\Omega_i$  onto  $Q$ , respectively  $T$ . It is obvious what the vertices and sides of  $\Omega_i$  correspond to. If  $\gamma$  is a side of  $\Omega_i$  then  $\tau_i$  induces mapping  $F_i$  of  $I$  onto  $\gamma$  (realizing that all the sides of the standard square and triangle have the same length as  $I$ ).

We shall assume the following about the partition and the mappings  $F_i$ :

a) If  $\bar{\Omega}_i \cap \bar{\Omega}_j = R_{i,j} \neq \emptyset$  then  $R_{i,j}$  is either a common vertex or a side of both  $\Omega_i$  and  $\Omega_j$ .

b) If  $R_{i,j} = \gamma_{i,j}$  then we will assume that the mappings  $F$  of  $I$  onto  $\gamma_{i,j}$  induced by the mappings  $\tau_i$  and  $\tau_j$  are identical. We denote  $F$  by  $F_{i,j}$ . This implies the following. Let  $A, B$  the vertices of  $\Omega_i$  and  $\Omega_j$  be the end points of  $\gamma_{i,j}$ . Assume that  $(a_1, b_1)$  and  $(a_2, b_2)$  are the end points of the sides  $\gamma_k^Q$  or  $\gamma_l^T$  such that  $\tau_i(a_1) = \tau_j(a_2) = A$ ,  $\tau_i(b_1) = \tau_j(b_2) = B$ . Then if  $C \in \gamma_{i,j}$ , and  $C = \tau_i(c_1) = \tau_j(c_2)$ ,  $\overline{a_1 c_1} = \overline{a_2 c_2}$  and  $\overline{c_1 b_1} = \overline{c_2 b_2}$ .

Since we assumed that  $\tau_i$  are smooth mappings, the vertices of  $\Omega$  necessarily have to coincide with some of the vertices of  $\Omega_i$ . We will further assume that for any  $\Gamma_j$  there is an element

$\Omega_1$  such that one of its sides coincides with  $\Gamma_j$ . This assumption is made without any loss of generality.

Denote now

$${}^1\mathcal{P}_p(\Omega) = \{u \in H^1(\Omega) \mid u|_{\Omega_1}(\Gamma_1(\ell)) \in \mathcal{P}_p^2(Q) \text{ if } \Omega_1 \text{ is a quadrilateral and } u|_{\Omega_1}(\Gamma_1(\ell)) \in \mathcal{P}_p^1(T) \text{ if } \Omega_1 \text{ is a triangle}\},$$

$${}^2\mathcal{P}_p(Q) = ({}^1\mathcal{P}_p(Q))^2.$$

$${}^2\mathcal{P}_p(\Gamma) = \{u \mid u_j|_{\Gamma_j}(\Gamma_1(\ell)) \in \mathcal{P}_p(I), j = 1, 2\}.$$

Here  $\Gamma = \Gamma_1$  or any side of an element. Let us define the constraint space  ${}^4\mathcal{P}_p(Q) = {}^2\mathcal{P}_p(Q)$  as follows.

i) If  $\Gamma_j$  is a total constraint side with end points  $A_j, A_{j+1}$  and  $u \in {}^4\mathcal{P}_p(Q)$ , then  $u_1(A_j) = u_1(A_{j+1}) = 0$ ,  $i = 1, 2$ , and

$$\int_{\Gamma_j} u_1 v_1 ds = 0, \text{ for all } v_1 \in {}^1\mathcal{P}_{p-2}(\Gamma_j), i = 1, 2.$$

ii) If  $\Gamma_j$  is a partial constraint side then

$$(\alpha^{(j)} u_1 + \beta^{(j)} u_2)(A_k) = 0, k = j, j+1$$

and

$$\int_{\Gamma_j} (\alpha^{(j)} u_1 + \beta^{(j)} u_2) v ds = 0, \text{ for all } v \in {}^2\mathcal{P}_{p-2}(\Gamma_j).$$

The  $\rho$  hyperplane  ${}^\rho\mathcal{P}_p(Q)$  is defined analogously. Let  $g$  be defined in terms of  $\rho$  by (2.8). Then on  $\Gamma_j$  we impose

$$u_i(A_k) = g_i^{(j)}(A_k), i = 1, 2, k = j, j+1$$

respectively

$$(\alpha^{(j)}u_1 + \beta^{(j)}u_2)(\Lambda_k) = g^{(j)}(\Lambda_k), \quad k = j, j+1$$

and

$$(2.9a) \quad \int_{\Gamma_j} u_i v_i ds = \int_{\Gamma_j} g_i^{(j)} v_i ds, \quad i = 1, 2$$

respectively

$$(2.9b) \quad \int_{\Gamma_j} (\alpha^{(j)}u_1 + \beta^{(j)}u_2) v ds = \int_{\Gamma_j} g^{(j)} v ds.$$

The p-version is then defined analogously as before. Find  $u_p \in \mathcal{X}_{\mathcal{P}_p}(\Omega)$  such that

$$(2.10) \quad B(u_p, v) = G_1(v) \quad \forall v \in \mathcal{X}_{\mathcal{P}_p}(\Omega).$$

**Remark.** Constraints of the type considered are typical in elasticity theory. Here  $u_1$  and  $u_2$  are the displacements in the directions  $x_1$  and  $x_2$  respectively. Assume now that the displacement is constrained in the normal direction only (and is friction free in the tangential direction). Then on the boundary, we obtain the partial constraint  $u_1 \cos \varphi + u_2 \sin \varphi = 0$  where  $\varphi$  is the angle of the outer normal with the axis  $x_1$ .

### 3. The convergence of the p-version of the finite element method.

#### 3.1. An abstract result

We will first describe an abstract framework which will be the basis for the forthcoming analysis.

Let  $X$  and  $W$  be Hilbert spaces and  $\tilde{X}_p \subset X$ ,  $\tilde{W}_p \subset W$ ,  $p = 1, 2, \dots$  be one parameter families of finite-dimensional subspaces.  $X_p \subset X$ ,  $W_p \subset W$  will denote corresponding families of hyperplanes such that  $(u-v) \in \tilde{X}_p$  whenever  $u, v \in X_p$  and  $(\phi-\psi) \in \tilde{W}_p$  whenever  $\phi, \psi \in W_p$ .

Let  $a(u, v)$ ,  $u, v \in X$  be a continuous bilinear form on  $X \times X$  and  $b(v, \phi)$  be a continuous bilinear form on  $X \times W$  such that  $b(v, \phi) \leq C \|v\|_X \|\phi\|_W$ .

Let  $u_0 \in X$ ,  $\phi_0 \in W$  and  $u_p \in X_p$ ,  $\phi_p \in W_p$  be such that

$$(3.1a) \quad a(u_0, v) + b(v, \phi_0) = F_1(v) \quad \forall v \in \tilde{X}_p$$

$$(3.1b) \quad b(u_0, \psi) = F_2(\psi) \quad \forall \psi \in \tilde{W}_p$$

and

$$(3.2a) \quad a(u_p, v) + b(v, \phi_p) = F_1(v) \quad \forall v \in \tilde{X}_p$$

$$(3.2b) \quad b(u_p, \psi) = F_2(\psi) \quad \forall \psi \in \tilde{W}_p.$$

Define  $Z_p = \{v \in \tilde{X}_p, b(v, \psi) = 0 \quad \forall \psi \in \tilde{W}_p\} \subset \tilde{X}_p$ . Then we have

**Theorem 3.1.** Let  $a(u, u) \geq \gamma \|u\|_X^2$ ,  $\gamma > 0$ , for any  $u \in Z_p$ . Then

$$(3.3) \quad \|u_0 - u_p\|_X \leq C \left[ \inf_{\substack{u_p - w_p \in \tilde{X}_p \\ b(u_0 - w_p, \psi) = 0 \quad \forall \psi \in \tilde{W}_p}} \|u_0 - w_p\|_X + \inf_{\phi_p \in W_p} \|\phi_0 - \phi_p\|_W \right]$$

(i.e.,  $u_p - w_p \in Z_p$ )

**Proof.** for arbitrary  $w_p \in X_p$  and  $\tau_p \in W_p$  we have

$$(3.4a) \quad a(u_p - w_p, v) + b(v, \varphi_p - \tau_p) = a(u_0 - w_p, v) + b(v, \varphi_0 - \tau_p) \quad \forall v \in \tilde{X}_p$$

$$(3.4b) \quad b(u_p - w_p, \varphi) = b(u_0 - w_p, \varphi) \quad \forall \varphi \in \tilde{W}_p$$

For  $v \in Z_p$  (3.4a) yields

$$(3.5) \quad a(u_p - w_p, v) = a(u_0 - w_p, v) + b(v, \varphi_0 - \tau_p)$$

Suppose now that  $w_p$  is such that

$$b(w_p, \varphi) = b(u_0, \varphi) \quad \forall \varphi \in \tilde{W}_p.$$

Then by (3.4b)

$$b(u_p - w_p, \varphi) = 0 \quad \forall \varphi \in \tilde{W}_p$$

and hence

$$u_p - w_p \in Z_p.$$

Now using  $v = u_p - w_p$  in (3.5) we get

$$a(u_p - w_p, u_p - w_p) \leq C[\|u_0 - w_p\|_X \|u_p - w_p\|_X + \|\varphi_0 - \tau_p\|_W \|u_p - w_p\|_X]$$

and hence by coercivity of  $a(\cdot, \cdot)$  on  $Z_p$ ,

$$\|u_p - w_p\|_X \leq C[\|u_0 - w_p\|_X + \|\varphi_0 - \tau_p\|_W]$$

and hence also

$$\|u_0 - w_p\|_X \leq C[\|u_0 - w_p\|_X + \|\varphi_0 - \tau_p\|_W]$$

from which (3.3) follows. □

### 3.2. The convergence of the p-version

Let  $u_0 = (u_{0,1}, u_{0,2}) \in {}^2H^1(\Omega)$  be the solution of our constrained problem (2.2), (2.5) and  $u_p = (u_{p,1}, u_{p,2}) \in {}^2P_p(\Omega)$  be the approximation given by (2.10).

We will assume that  $u_0 \in {}^2H^k(\Omega)$ ,  $k > 3/2$ . Hence  $\frac{\partial u_0}{\partial n} \in {}^2H^0(\Gamma_j)$ ,  $j = 1, 2, \dots, m$ . Let the constraints on  $\Gamma_j$  be as in section 2.2, with  $\Gamma_{1j}$ ,  $j = 1, \dots, m_1$  being the total constraint sides and  $\Gamma_{1j}$ ,  $j = m_1+1, \dots, m$  the partial constraint sides. Then it can be verified that for any  $v \in {}^2H^1(\Omega)$ ,

$$\begin{aligned} B(u_0, v) &= \sum_{j=1}^{m_1} \int_{\Gamma_{1j}} \left[ \frac{\partial u_{0,1}}{\partial n} v_1 + \frac{\partial u_{0,2}}{\partial n} v_2 \right] ds \\ (3.6) \quad &= \sum_{j=m_1+1}^m \int_{\Gamma_{1j}} (\alpha^{1j} v_1 + \beta^{1j} v_2) \left[ \frac{\partial u_{0,1}}{\partial n} + \frac{\partial u_{0,2}}{\partial n} \right] ds = G_1(v) \end{aligned}$$

where we have assumed  $\alpha^{1j} + \beta^{1j} = 1$ . Moreover, for  $v, v_1, v_2$  and  $\rho$  as in (2.8) - (2.9),

$$\begin{aligned} &\sum_{j=1}^{m_1} \int_{\Gamma_{1j}} (u_{p,1} v_1 + u_{p,2} v_2) ds + \sum_{j=m_1+1}^m \int_{\Gamma_{1j}} (\alpha^{1j} u_{p,1} + \beta^{1j} u_{p,2}) v ds \\ (3.7) \quad &= \sum_{j=1}^{m_1} \int_{\Gamma_{1j}} (u_{0,1} v_1 + u_{0,2} v_2) ds + \sum_{j=m_1+1}^m \int_{\Gamma_{1j}} (\alpha^{1j} u_{0,1} + \beta^{1j} u_{0,2}) v ds \\ &= \sum_{j=1}^{m_1} \int_{\Gamma_{1j}} (\rho_1 v_1 + \rho_2 v_2) ds + \sum_{j=m_1+1}^m \int_{\Gamma_{1j}} (\alpha^{1j} \rho_1 + \beta^{1j} \rho_2) v ds \end{aligned}$$

and

$$\begin{aligned} (3.8a) \quad u_{p,k}(A_j) &= u_{0,k}(A_j) = \rho_k(A_j), \quad j = \ell, \ell+1, \quad k = 1, 2, \\ &\ell = 1, \dots, 1_{m_1} \end{aligned}$$

$$\begin{aligned}
(\alpha^\ell u_{p,1})(A_j) + (\beta^\ell u_{p,2})(A_j) &= (\alpha^\ell u_{0,1})(A_j) + (\beta^\ell u_{0,2})(A_j) \\
(3.8b) \qquad \qquad \qquad &= (\alpha^\ell \rho_1)(A_j) + (\beta^\ell \rho_2)(A_j) \\
&\qquad \qquad \qquad j = \ell, \ell+1, \ell = i_{m_1+1}, \dots, i_m.
\end{aligned}$$

(We remark that  $u_{0,i}(A_j)$ ,  $i = 1, 2$  has meaning because we assumed that  $u_0 \in {}^2H^k(Q)$ ,  $k > 3/2$ ).

We now define

$$X = {}^2H^1(Q), \quad \|\cdot\|_X = \|\cdot\|_{{}^2H^1(Q)}$$

and for any  $\delta = (\delta_1, \delta_2) \in X$ ,

$$\begin{aligned}
X_{p,\delta} &= \{u = (u_1, u_2) \in {}^2\mathcal{P}_p(Q), u_k(A_j) = \delta_k(A_j), k = 1, 2 \\
&\quad j = \ell, \ell+1, \ell = i_1, \dots, i_{m_1}, (\alpha^\ell u_1)(A_j) + (\beta^\ell u_2)(A_j) \\
&\quad = (\alpha^\ell \delta_1)(A_j) + (\beta^\ell \delta_2)(A_j), j = \ell, \ell+1, \ell = i_{m_1+1}, \dots, i_m\}.
\end{aligned}$$

We then take in our abstract framework

$$X_p = X_{p,\rho}, \quad \tilde{X}_p = X_{p,0}$$

where  $\rho$  satisfies (2.6) - (2.8). Moreover, let

$$W = \prod_{j=1}^{m_1} {}^2H^{-1/2}(\Gamma_{1j}) \cdot \prod_{j=m_1+1}^m {}^1H^{-1/2}(\Gamma_{1j})$$

with the norm

$$\|\cdot\|_W = \left[ \sum_{j=1}^{m_1} \|\cdot\|_{{}^2H^{-1/2}(\Gamma_{1j})}^2 + \sum_{j=m_1+1}^m \|\cdot\|_{{}^1H^{-1/2}(\Gamma_{1j})}^2 \right]^{1/2}.$$

We see then that  $\phi_0 \in W$  where (see (3.6))

$$\begin{aligned}\phi_{0,j} &= - \left[ \frac{\partial u_{0,1}}{\partial n} \Big|_{\Gamma_{1j}}, \frac{\partial u_{0,2}}{\partial n} \Big|_{\Gamma_{1j}} \right], \quad j = 1, \dots, m_1 \\ &= - \left[ \frac{\partial u_{0,1}}{\partial n} + \frac{\partial u_{0,2}}{\partial n} \right] \Big|_{\Gamma_{1j}}, \quad j = m_1+1, \dots, m.\end{aligned}$$

Define  $W_p = \tilde{W}_p = \prod_{j=1}^m {}^j W_p \subset W$  where

$$\begin{aligned}{}^j W_p &= {}^2 H^{-1/2}(\Gamma_{1j}) \cap {}^2 \mathcal{P}_{p-2}(\Gamma_j), \quad j = 1, \dots, m_1 \\ &= {}^1 H^{-1/2}(\Gamma_{1j}) \cap {}^1 \mathcal{P}_{p-2}(\Gamma_j), \quad j = m_1+1, \dots, m.\end{aligned}$$

Let  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$  be bilinear forms defined respectively on  $X \times X$  and  $X \times W$  by

$$a(u, v) = B(u, v)$$

$$b(u, v) = \sum_{j=1}^{m_1} \int_{\Gamma_{1j}} (u_1 v_1 + u_2 v_2) ds + \sum_{j=m_1+1}^m \int_{\Gamma_{1j}} (\alpha {}^1 j u_1 + \beta {}^1 j u_2) v ds.$$

It may be seen that the right hand side of (3.7) defines a linear functional  $G_2$  on  $W_p$ . Then (3.6) - (3.8) show that  $(u_0, \phi_0)$  satisfy (3.1) with  $F_k = G_k$ ,  $k = 1, 2$ . Moreover, if we can find a unique pair  $(u_p, \phi_p)$  satisfying (3.2), then  $u_p$  will be precisely our finite element solution satisfying (2.10). We will now verify that the mixed method defined above satisfies the assumptions of Theorem 3.1. This in turn will lead to the existence and uniqueness of the solution  $(u_p, \phi_p)$  of (3.2) and an estimate of the rate of convergence of  $u_p$  to  $u$ .

Obviously,  $a(u, v)$  satisfies the desired continuity and



coercivity conditions. For  $j = 1, 2, \dots, m$  we have

$$\left| \int_{\Gamma_{1j}} u v \, ds \right| \leq C u_{H^{1/2}(\Gamma_{1j})} v_{H^{-1/2}(\Gamma_{1j})}$$

from which the continuity of  $b(\cdot, \cdot)$  may be deduced. Hence

Theorem 3.1 is applicable. Let us now estimate  $\inf_{r \in W_p} \varphi_0 - r W$ .

First, let  $m_1 + 1 \leq j \leq m$ . We assumed that  $\varphi_{0,j} \in H^{k-3/2}(\Gamma_{1j})$ ,  $k >$

$3/2$ . Hence  $\varphi_{0,j}(F_{1j}(t)) = \phi(t) \in H^{k-3/2}(I)$ . Let  $\sigma \in {}^1P_{p-2}(I)$

be such that

$$(3.9) \quad \int_I \sigma \rho \, dt = \int_I \phi \rho \, dt, \quad \forall \rho \in {}^1P_{p-2}(I).$$

Then, with  $q = \phi - \sigma$ , we have

$$(3.10) \quad q_{H^0(I)} \leq C p^{-(k-3/2)} \phi_{H^{k-3/2}(I)}.$$

Now, for arbitrary  $v \in H^1(I)$ , we have by (3.9)

$$\frac{\int_I q v \, dt}{v_{H^1(I)}} = \frac{\int_I q(v - \sigma_1) \, dt}{v_{H^1(I)}} \leq \frac{q_{H^0(I)} v_{H^0(I)} - \sigma_1_{H^0(I)}}{v_{H^1(I)}} \leq C p^{-1} q_{H^0(I)}$$

where  $\sigma_1$  is a polynomial of degree  $p - 2$  satisfying

$$v - \sigma_1_{H^0(I)} = C p^{-1} v_{H^1(I)}.$$

This yields

$$(3.11) \quad q_{H^{-1}(I)} \leq C p^{-(k-1/2)} \phi_{H^{k-3/2}(I)}.$$

Interpolating (3.10), (3.11) and using the fact that  $F_{1j}$  is a

smooth mapping, we obtain

$$\inf_{\chi_p \in W_p} \|\phi_{0,j} - \chi_p\|_{H^{-1/2}(\Gamma_{1j})} \leq C p^{-(k-1)} \|\phi_{0,j}\|_{H^{k-3/2}(\Gamma_{1j})}.$$

We get similar estimates for  $\Gamma_{1j}$ ,  $j = 1, \dots, m_1$ , so that

$$(3.12) \quad \inf_{\chi_p \in W_p} \|\phi_0 - \chi_p\|_W \leq C p^{-(k-1)} \|u_0\|_{2, H^k(Q)}.$$

We now estimate  $\inf \|u_0 - w_p\|_X$ . Using the results from [2], there exist  $z_i \in {}^1\mathcal{P}_p(Q)$ ,  $i = 1, 2$  such that

$$(3.13a) \quad \|u_{0,i} - z_i\|_{H^t(Q)} \leq C p^{-(k-t)} \|u_{0,i}\|_{H^k(Q)}, \quad i = 1, 2, \quad t = 0, 1$$

$$(3.13b) \quad u_{0,i}(N) = z_i(N) \quad \text{for each node } N \text{ of the mesh}$$

$$(3.13c) \quad \|u_{0,i} - z_i\|_{H^t(\Gamma_j)} \leq C p^{-(k-1/2-t)} \|u_{0,i}\|_{H^k(Q)}, \quad t = 0, 1, \quad i = 1, 2 \\ j = 1, 2, \dots, m$$

Let  $m_1 + 1 \leq j \leq m$ . Let us denote  $\kappa = \alpha^{1j}(u_{0,1} - z_1) + \beta^{1j}(u_{0,2} - z_2)$ . Then we have  $\kappa(A_{1j}) = \kappa(A_{1j+1}) = 0$ . Let  $\tilde{\kappa}(\xi) = \kappa(F(\xi))$  and let  $\tau(F(\xi)) = \tilde{\tau}(\xi) \in {}^1\mathcal{P}_p(I)$  satisfy  $\tau(\pm 1) = 0$  and

$$\int_I \tilde{\tau} v \, d\xi = \int_I \tilde{\kappa} v \, d\xi, \quad \forall v \in {}^1\mathcal{P}_{p-2}(I)$$

Because of (3.13b) we can write

$$\int_I \tilde{\tau}' \omega' \, d\xi = \int_I \tilde{\kappa}' \omega' \, d\xi, \quad \forall \omega \in {}^1\mathcal{P}_p(I), \quad \omega(\pm 1) = 0$$

and hence by Lemma 3.2 of [2]

$$\|\tilde{\kappa} - \tilde{\tau}\|_{H^t(I)} \leq C p^{-(k-1/2-t)} \|u_0\|_{2, H^k(Q)}, \quad t = 0, 1.$$

Using (3.13c), this gives

$$\|\tilde{r}\|_{H^t(I)} \leq C p^{-(k-1/2-t)} \|u_0\|_{H^k(\Omega)}, \quad t = 0, 1.$$

Now using Lemma 2.1 it follows that there is a  $w \in {}^1\mathcal{P}_p(\Omega)$  such that  $w = 0$  on  $\Omega - \tilde{\Omega}$  where  $\tilde{\Omega}$  is the element with the side  $\Gamma_{i_j}$ ,  $w = \tau$  on  $\Gamma_{i_j}$ ,  $w = 0$  on  $\partial\tilde{\Omega} - \Gamma_{i_j}$  and

$$(3.14) \quad \|w\|_{H^1(\Omega)} \leq C p^{-(k-1)} \|u_0\|_{H^k(\Omega)}.$$

Letting  $w_{i_j} = (w, w) \in {}^2H^1(\Omega)$ , we see that  $w_{i_j}$  will satisfy (3.14) with  $H^1(\Omega)$  replaced by  ${}^2H^1(\Omega)$ .

Let now  $w_p = z + w_{i_j} = (z_1, z_2) + (w, w)$ . Using (3.13b) and the fact that  $w(A_\ell) = 0$ , we obtain

$$(\alpha^{i_j}(u_{0,1} - w_{p,1}) + \beta^{i_j}(u_{0,2} - w_{p,2}))(A_\ell) = 0, \quad \ell = i_j, i_j+1.$$

Moreover, for  $\psi \in {}^1\mathcal{P}_{p-2}(\Gamma_{i_j})$ ,

$$\begin{aligned} & \int_{\Gamma_{i_j}} (\alpha^{i_j}(u_{0,1} - w_{p,1}) + \beta^{i_j}(u_{0,2} - w_{p,2})) \psi \, ds \\ &= \int_{\Gamma_{i_j}} (\kappa - (\alpha^{i_j} + \beta^{i_j})w) \psi \, ds = \int_{\Gamma_{i_j}} (\kappa - w) \psi \, ds = 0 \end{aligned}$$

where we have used  $\alpha^{i_j} + \beta^{i_j} = 1$ . We may construct  $w_{i_j}$  as above for all partial constraint sides. An analogous construction can be carried out for total constraint sides as well. Then if

$$w_p = z + \sum_{j=1}^m w_{i_j}$$

we see that

$$u_p - w_p \in \tilde{X}_p$$

$$b(u_0 - w_p, \psi) = 0, \quad \forall \psi \in W_p$$

and

$$\|u_0 - w_p\|_X \leq \|u_0 - z\|_X + \sum_{j=1}^m \|w_{1j}\|_X \leq Cp^{-(k-1)} \|u_0\|_{H^k(\Omega)}.$$

This provides a bound for the first term in the right hand side of (3.3). Hence we have proven

Theorem 3.2. Let  $u_0 \in H^k(\Omega)$ ,  $k > 3/2$ . Then

$$\|u_0 - u_p\|_{H^1(\Omega)} \leq Cp^{-(k-1)} \|u_0\|_{H^k(\Omega)}$$

where  $u_0$  is the exact solution and  $u_p$  is the finite element solution of the constrained problem, provided that  $u_0$  and  $u_p$  exist.

The next theorem deals with the question of existence and uniqueness of  $(u_0, \phi_0)$  and  $(u_p, \phi_p)$ .

Theorem 3.3. The (exact) solution  $(u_0, \phi_0)$  of the constrained problem exists. The finite element solution  $(u_p, \phi_p)$  exists and is unique.

Proof. In Section 2.2 we have shown that  $u_0$  exists and hence  $(u_0, \phi_0)$  exists, too. The finite element solution  $(u_p, \phi_p)$  is determined by the solution of a linear system of equations with square matrix. Hence the existence follows from the uniqueness. Assume therefore that there is a solution  $(u_p, \phi_p)$  of the trivial problem. Obviously  $\bar{u} = \bar{\phi} = 0$  is also a solution of this problem. Hence  $u_p = 0$  because of Theorem 3.1. We have to show therefore that

$$\int_I (\alpha v_1 + \beta v_2) \varphi_p \, d\xi = 0$$

implies  $\varphi_p = 0$ . Because  $\alpha + \beta = 1$  we also have  $\int_I v \varphi_p \, d\xi = 0$  for all  $v \in {}^1\mathcal{P}_p(I) \cap H_0^1(I)$  while  $\varphi_p \in {}^1\mathcal{P}_{p-2}(I)$ . This leads to  $\varphi_p = 0$  which leads to the desired result.  $\square$

Remark. We have dealt only with a model problem. It is obvious that the theorem holds in general, as for example, for the theory of elasticity.

#### 4. Some aspects of implementation.

Here we will make some comments about the implementation in the framework of the code PROBE\* (see [6]). The shape functions are defined as usual on the standard square or triangle. There are three types:

- a) the model shape functions which are linear on every side of  $Q$ , respectively  $T$ ;
- b) the side functions which are zero at the vertices of  $Q$ , respectively  $T$  and on  $\gamma$  are of the form

$$\xi_j = \int_{-1}^X \ell_j(\xi) dx, \quad j = 1, 2, \dots$$

where  $\ell_j$  is the Legendre polynomial of degree  $j$ .  $\xi_j$  is then a polynomial of degree  $j+1$ ;

- c) The internal shape functions which are zero on  $\partial Q$  (respectively  $\partial T$ ).

The stiffness matrices are first computed in the standard way without constraints. Then the constraints are imposed at the vertices  $A_j$ . This only involves the amplitudes for the nodal shape functions. Then the conditions (2.9 a,b) only involve amplitudes for the side shape functions. The functions  $\psi$  in (2.9 a,b) are computed as derivatives of the Legendre polynomials from the usual recurrence formula and the integration is made using numerical quadrature.

The condition (2.9a) is especially simple because  $u_1 = \sum c_1 \xi_1$ . Integrating by parts and exploiting orthogonality of the

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\* The code PROBE is the code of Noetic Tech., St. Louis.

Legendre polynomials we get the amplitudes for the side shape functions on the total constraint sides directly.

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